COMMON FIXED POINTS FOR THREE OR FOUR MAPPINGS VIA COMMON FIXED POINT FOR TWO MAPPINGS

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Abstract. In this paper, we shall show that some coincidence point and common fixed point results for three or four mappings could easily be obtained from the corresponding fixed point results for two mappings.

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1. Introduction and Preliminaries

Fixed point theory is an important and actual topic of nonlinear analysis [1-7, 9-12]. Moreover, it's well known that the contraction mapping principle, formulated and proved in the Ph.D. dissertation of Banach in 1920 which was published in 1922 is one of the most important theorems in classical functional analysis.

It is also known that common fixed point theorems are generalizations of fixed point theorems. Over the past few decades, there have many researchers interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems. Recently, Haghi et al. [8] have shown that some coincidence point and common fixed point results for two mappings are not real generalizations as they could easily be obtained from the corresponding fixed point theorems. They used the following Lemma that is a consequence of the axiom of choise.

Lemma 1 ([8], Lemma 2.1)**.** *Let X be a nonempty set and $f : X \to X$ a function. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one.*

In this paper, we shall show that some coincidence point and common fixed point for three or four mappings could easily be obtained from the corresponding fixed point results for two mappings.

2. Main results

In this section, first we prove a result of common fixed point for two self-mappings satisfying a contractive condition of Berinde type [3]. Successively, we deduce some common fixed point results for three or four self-mappings.

Theorem 1. *Let (X, d) be a metric space and S and T self-mappings on X. Assume that the following condition holds:*

$$\begin{aligned} d(Sx, Ty) \leq{} & \alpha d(x, Sx) + \beta d(y, Ty) + \gamma d(x, y) + \delta[d(y, Sx) + d(x, Ty)] \\ & + L \min\{d(x, Sx), d(y, Ty), d(y, Sx), d(x, Ty)\} \end{aligned} \tag{1}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta \in [0, 1[$ are such that $\alpha + \beta + \gamma + 2\delta < 1$ and $L \geq 0$. If SX or TX is a complete subspace of X, then S and T have a unique common fixed point in X.

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Proof. Let $x_0 \in X$. We construct the sequence $\{x_n\}$ such that

$$x_1 = Sx_0,\ x_2 = Tx_1, \dots, x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1}, \dots .$$

We show that $\{x_n\}$ is a Cauchy sequence in X. Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (1), we obtain

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) =& d(Sx_{2n}, Tx_{2n+1}) \le \alpha d(x_{2n+1}, x_{2n}) + \beta d(x_{2n+1}, x_{2n+2}) \\ & + \gamma d(x_{2n}, x_{2n+1}) + \delta[d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+2})] \\ & + L \min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+2})\}. \end{aligned}$$

Consequently, we have

$$(1 - \beta - \delta) d(x_{2n+1}, x_{2n+2}) \le (\alpha + \gamma + \delta) d(x_{2n}, x_{2n+1}).$$

Similarly, Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (1), we get

$$(1 - \alpha - \delta) d(x_{2n+3}, x_{2n+2}) \le (\beta + \gamma + \delta) d(x_{2n+1}, x_{2n+2}).$$

Let $k = \max\{\frac{\alpha+\gamma+\delta}{1-\beta-\delta}, \frac{\beta+\gamma+\delta}{1-\alpha-\delta}\} < 1$. Then, we have

$$d(x_{n+1}, x_n) \le k d(x_n, x_{n-1}) \le \cdots \le k^n d(x_1, x_0).$$

It implies that $\{x_n\}$ is a Cauchy sequence. If SX is complete, then $x_{2n-1} \to z \in SX$ and hence $x_n \to z$. (The same holds if TX is complete with $z \in TX$.) Now, we prove that $z = Sz$. If not, we have

$$\begin{aligned} d(Sz, z) &\le d(Sz, Tx_{2n+1}) + d(Tx_{2n+1}, z) \\ &\le d(x_{2n+2}, z) + \alpha d(Sz, z) + \beta d(x_{2n+1}, x_{2n+2}) + \gamma d(x_{2n+1}, z) \\ &\quad + \delta[d(Sz, x_{2n+1}) + d(x_{2n+2}, z)] \\ &\quad + L \min\{d(Sz, z), d(x_{2n+1}, x_{2n+2}), d(Sz, x_{2n+1}), d(x_{2n+2}, z)\}, \end{aligned}$$

as $n \to +\infty$, we get

$$d(Sz, z) \le (\alpha + \delta) d(Sz, z).$$

This is a contradiction and hence $z = Sz$.

Similarly, we deduce that $z = Tz$ and so z is a common fixed point of S and T. The uniqueness follows by (1). □

Remark 1. Clearly, Theorem 1 holds if we assume that X is a complete metric space.

Let X be a non-empty set and $T, f : X \to X$. The mappings T, f are said to be weakly compatible if they commute at their coincidence point (i.e. $Tfx = fTx$ whenever $Tx = fx$). A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$.

Lemma 2. *Let X be a non-empty set and the mappings $T,\ f : X \to X$ have a unique point of coincidence v in X. If T and f are weakly compatible, then T and f have a unique common fixed point.*

Proof. Since v is a point of coincidence of T and f. Therefore, $v = fu = Tu$ for some $u \in X$. By weakly compatibility of T and f we have

$$Tv = Tfu = fTu = fv.$$

It implies that $Tv = fv = w$ (say). Then w is a point of coincidence of T and f. Therefore, $v = w$ by uniqueness. Thus v is a unique common fixed point of T and f. □

From Theorem 1, we deduce the following theorems.

Theorem 2. *Let (X,d) be a metric space and S,T and f self-mappings on X such that $SX \cup TX \subset fX$. Assume that the following condition holds:*

$$
\begin{aligned}
d(Sx,Ty) \leq{} & \alpha d(fx,Sx) + \beta d(fy,Ty) + \gamma d(fx,fy) \\
& + \delta[d(fy,Sx) + d(fx,Ty)] \\
& + L\min\{d(fx,Sx), d(fy,Ty), d(fy,Sx), d(fx,Ty)\},
\end{aligned} \tag{2}
$$

for all $x,y \in X$, where $\alpha,\beta,\gamma,\delta \in [0,1[$ are such that $\alpha+\beta+\gamma+2\delta < 1$ and $L \geq 0$. If (S,f) and (T,f) are weakly compatible and fX is a complete subspace of X, then S,T and f have a unique common fixed point in X.

Proof. By Lemma 1 there exists $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one. Now, define two mappings $g,h : fE \to fE$ by $g(fx) = Sx$ and $h(fx) = Tx$, respectively. Since f is one-to-one on E, then g,h are well-defined. Note that,

$$
\begin{aligned}
d(g(fx),h(fy)) \leq{} & \alpha d(fx,g(fx)) + \beta d(fy,h(fy)) + \gamma d(fx,fy) \\
& + \delta[d(fy,g(fx)) + d(fx,h(fy))] \\
& + L\min\{d(fx,g(fx)), d(fy,h(fy)), d(fy,g(fx)), d(fx,h(fy))\}.
\end{aligned} \tag{3}
$$

By Theorem 1, as fE is a complete subspace of X, we deduce that there exists a unique common fixed point $fz \in fE$ of g and h, that is $fz = g(fz) = h(fz)$. Thus z is a coincidence point of S,T and f. Note that, if $Sx = Tx = fx$, using (2) we get $fx = fz$ and so S,T and f have a unique point of coincidence $fz \in fE$. Now, since (S,f) and (T,f) are weakly compatible, by Lemma 2, we deduce that fz is a unique fixed point of S,T and f. □

Theorem 3. *Let (X,d) be a metric space and S,T,f and g self-mappings on X such that $SX,\, TX \subset fX = gX$. Assume that the following condition holds:*

$$
\begin{aligned}
d(Sx,Ty) \leq{} & \alpha d(fx,Sx) + \beta d(gy,Ty) + \gamma d(fx,gy) + \delta[d(gy,Sx) + d(fx,Ty)] \\
& + L\min\{d(fx,Sx), d(gy,Ty), d(gy,Sx), d(fx,Ty)\},
\end{aligned} \tag{4}
$$

for all $x,y \in X$, where $\alpha,\beta,\gamma,\delta \in [0,1[$ are such that $\alpha+\beta+\gamma+2\delta < 1$ and $L \geq 0$. If (S,f) and (T,g) are weakly compatible and fX is a complete subspace of X, then S,T,f and g have a unique common fixed point in X.

Proof. By Lemma 1, there exists $E_1, E_2 \subset X$ such that $fE_1 = fX = gX = gE_2$, $f : E_1 \to X$ and $g : E_2 \to X$ are one-to-one. Now, define two mappings $A,B : fE_1 \to fE_1$ by $A(fx) = Sx$ and $B(gx) = Tx$, respectively. Since f,g are one-to-one on E_1, E_2, respectively, then the mappings A,B are well-defined. Note that,

$$
\begin{aligned}
d(Afx,Bgy) \leq{} & \alpha d(fx,Afx) + \beta d(gy,Bgy) + \gamma d(fx,gy) + \delta[d(gy,Afx) + d(fx,Bgy)] \\
& + L\min\{d(fx,Afx), d(gy,Bgy), d(gy,Afx), d(fx,Bgy)\},
\end{aligned}
$$

for all $fx, gy \in fE_1$.

By Theorem 1, as fE_1 is a complete subspace of X, we deduce that there exists a unique common fixed point $fz \in fE_1$ of A and B. Thus z is a coincidence point of S and f, that is $Sz = fz$. Now, let $v \in X$ such that $fz = gv$, then v is a coincidence point of T and g, that is $Tv = gv$. We show that S and f have a unique point of coincidence. If $Sw = fw$ and $fw \neq fz$, from (4) for $x = w$ and $y = v$, we get

$$d(fw,gv) = d(Sw,Tv) \leq (\gamma + 2\delta)d(fw,gv),$$

which implies $fw = gv = fz$. Since S and f are weakly compatible, by Lemma 2, it follows that fz is the unique common fixed point of S and f. Similarly, gv is the unique common fixed point for T and g and so $fz = gv$ is the unique common fixed point for S, T, f and g. □

From the proof of the Theorems 2-3 it follows that the following results of points of coincidence for three or four self-mappings hold.

Theorem 4. *Let (X, d) be a metric space and S, T and f self-mappings on X such that $SX \cup TX \subset fX$. Assume that the following condition holds:*

$$\begin{aligned} d(Sx, Ty) \leq & \alpha d(fx, Sx) + \beta d(fy, Ty) + \gamma d(fx, fy) \\ & + \delta[d(fy, Sx) + d(fx, Ty)] \\ & + L \min\{d(fx, Sx), d(fy, Ty), d(fy, Sx), d(fx, Ty)\}, \end{aligned} \tag{5}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta \in [0, 1[$ are such that $\alpha + \beta + \gamma + 2\delta < 1$ and $L \geq 0$. If fX is a complete subspace of X, then S, T and f have a unique point of coincidence in X.

Theorem 5. *Let (X, d) be a metric space and S, T, f and g self-mappings on X such that SX, $TX \subset fX = gX$. Assume that the following condition holds:*

$$\begin{aligned} d(Sx, Ty) \leq & \alpha d(fx, Sx) + \beta d(gy, Ty) + \gamma d(fx, gy) + \delta[d(gy, Sx) + d(fx, Ty)] \\ & + L \min\{d(fx, Sx), d(gy, Ty), d(gy, Sx), d(fx, Ty)\}, \end{aligned} \tag{6}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta \in [0, 1[$ are such that $\alpha + \beta + \gamma + 2\delta < 1$ and $L \geq 0$. If fX is a complete subspace of X, then S, T, f and g have a unique point of coincidence in X.

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